

1.

$$y'' + 8y' + 25y = 10x + e^{-4x} \sin 3x$$

The complementary solution $y_c(x)$:

Solving the homogenous ODE:

$$y'' + 8y' + 25y = 0$$

Writing the characteristic equation:

$$r^2 + 8r + 25 = 0$$

$$r = \frac{-8 \pm \sqrt{8^2 - 4(1)(25)}}{2(1)}$$

$$r = -4 \pm 3i$$

Then the complementary solution is:

$$y_c(x) = c_1 e^{-4x} \cos 3x + c_2 e^{-4x} \sin 3x$$

The particular solution $y_p(x)$:

Using the method of undetermined coefficients:

Guesses for the form of the solution:

For $10x$: $Ax + B$

For e^{-4x} : Cxe^{-4x}

(multiplied by x since e^{-4x} is already present in the complementary solution)

For $\sin 3x$: $D \sin 3x + E \cos 3x$

Putting these together:

$$y_p(x) = Ax + B + Cxe^{-4x}(D \sin 3x + E \cos 3x)$$

$$y_p(x) = Ax + B + xe^{-4x}(CD \sin 3x + CE \cos 3x)$$

Considering the product of constants constant:

$$y_p(x) = Ax + B + xe^{-4x}(C \sin 3x + D \cos 3x)$$

Differentiating:

$$y_p'(x) = A + e^{-4x}(C \sin 3x + D \cos 3x) - 4xe^{-4x}(C \sin 3x + D \cos 3x) \\ + xe^{-4x}(3C \cos 3x - 3D \sin 3x)$$

$$y_p'(x) = A + e^{-4x}(C \sin 3x + D \cos 3x) - xe^{-4x}((4C + 3D) \sin 3x + (4D - 3C) \cos 3x)$$

$$y_p''(x) = [e^{-4x}(3C \cos 3x - 3D \sin 3x) - 4e^{-4x}(C \sin 3x + D \cos 3x)] \\ - [e^{-4x}((4C + 3D) \sin 3x + (4D - 3C) \cos 3x) \\ - 4xe^{-4x}((4C + 3D) \sin 3x + (4D - 3C) \cos 3x) \\ + xe^{-4x}((12C + 9D) \cos 3x - (12D - 9C) \sin 3x)]$$

$$\begin{aligned}
y_p''(x) = & -e^{-4x}((4C + 3D) \sin 3x + (4D - 3C) \cos 3x) \\
& - e^{-4x}((4C + 3D) \sin 3x + (4D - 3C) \cos 3x) \\
& + 4xe^{-4x}((4C + 3D) \sin 3x + (4D - 3C) \cos 3x) \\
& - xe^{-4x}((12C + 9D) \cos 3x - (12D - 9C) \sin 3x)
\end{aligned}$$

$$\begin{aligned}
y_p''(x) = & xe^{-4x}((7C + 24D) \sin 3x + (7D - 24C) \cos 3x) \\
& - e^{-4x}((8C + 6D) \sin 3x + (8D - 6C) \cos 3x)
\end{aligned}$$

Since $y_p(x)$ is a solution to the given nonhomogeneous ODE, we can plug in $y_p(x)$ and its derivatives:

$$\begin{aligned}
& [xe^{-4x}((7C + 24D) \sin 3x + (7D - 24C) \cos 3x) \\
& \quad - e^{-4x}((8C + 6D) \sin 3x + (8D - 6C) \cos 3x)] \\
& \quad + 8[A + e^{-4x}(C \sin 3x + D \cos 3x) \\
& \quad - xe^{-4x}((4C + 3D) \sin 3x + (4D - 3C) \cos 3x)] \\
& \quad + 25[Ax + B + xe^{-4x}(C \sin 3x + D \cos 3x)] = 10x + e^{-4x} \sin 3x \\
25Ax + 25B + 8A + xe^{-4x}(0 \cdot \sin 3x + 0 \cdot \cos 3x) + e^{-4x}(-6D \sin 3x + 6C \cos 3x) \\
& = 10x + e^{-4x} \sin 3x
\end{aligned}$$

$$25Ax + 25B + 8A - 6De^{-4x} \sin 3x + 6Ce^{-4x} \cos 3x = 10x + e^{-4x} \sin 3x$$

$$25A = 10 \rightarrow A = \frac{2}{5}$$

$$8A + 25B = 0 \rightarrow B = -\frac{16}{125}$$

$$-6D = 1 \rightarrow D = -\frac{1}{6}$$

$$6C = 0 \rightarrow C = 0$$

Plugging in the values of the coefficients in $y_p(x)$:

$$y_p(x) = \frac{2}{5}x - \frac{1}{6}xe^{-4x} \cos 3x - \frac{16}{125}$$

The general solution is:

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 e^{-4x} \cos 3x + c_2 e^{-4x} \sin 3x + \frac{2}{5}x - \frac{1}{6}xe^{-4x} \cos 3x - \frac{16}{125}$$

Solving the initial value problem:

Using the conditions $y(0) = 0$ and $y'(0) = 0$:

$$y(0) = c_1 e^{-4(0)} \cos 3(0) + c_2 e^{-4(0)} \sin 3(0) + \frac{2}{5}(0) - \frac{1}{6}(0)e^{-4(0)} \cos 3(0) - \frac{16}{125} = 0$$

$$c_1 - \frac{16}{125} = 0 \rightarrow c_1 = \frac{16}{125}$$

$$\begin{aligned}
y'(x) = & c_1 e^{-4x}(-4 \cos 3x - 3 \sin 3x) + c_2 e^{-4x}(-4 \sin 3x + 3 \cos 3x) + \frac{2}{5} \\
& + \frac{1}{6}e^{-4x}(\cos 3x - 4x \cos 3x - 3x \sin 3x)
\end{aligned}$$

$$y'(0) = c_1 e^{-4(0)}(-4 \cos 3(0) - 3 \sin 3(0)) + c_2 e^{-4(0)}(-4 \sin 3(0) + 3 \cos 3(0)) + \frac{2}{5} \\ + \frac{1}{6} e^{-4(0)}(\cos 3(0) - 4(0) \cos 3(0) - 3(0) \sin 3(0)) = 0$$

$$-4c_1 + 3c_2 + \frac{2}{5} - \frac{1}{6} = 0 \rightarrow c_2 = \frac{209}{2250}$$

$$y(x) = \frac{16}{125} e^{-4x} \cos 3x + \frac{209}{2250} e^{-4x} \sin 3x + \frac{2}{5} x - \frac{1}{6} x e^{-4x} \cos 3x - \frac{16}{125}$$

2.

(1)

$$(x \sin x + \cos x)y'' - (x \cos x)y' + (\cos x)y = 0$$

Normalizing the first coefficient:

$$y'' - \frac{x \cos x}{x \sin x + \cos x} y' + \frac{\cos x}{x \sin x + \cos x} y = 0$$

By Abel's Theorem:

$$W = k e^{-\int -\frac{x \cos x}{x \sin x + \cos x} dx}$$

$$= \int -\frac{x \cos x}{x \sin x + \cos x} dx$$

$$= \int \frac{x \cos x}{x \sin x + \cos x} dx$$

$$u = x \sin x + \cos x$$

$$\frac{du}{dx} = (\sin x + x \cos x) - \sin x = x \cos x$$

$$dx = \frac{du}{x \cos x}$$

$$= \int \frac{x \cos x}{u} \frac{du}{x \cos x}$$

$$= \int \frac{1}{u} du$$

$$= \ln(x \sin x + \cos x)$$

$$W = k e^{\ln(x \sin x + \cos x)}$$

$$\mathbf{W = kx \sin x + k \cos x}$$

(2)

Substituting $y_1(x) = x$:

$$y = x$$

$$y' = 1$$

$$y'' = 0$$

$$0 - \frac{x \cos x}{x \sin x + \cos x} (1) + \frac{\cos x}{x \sin x + \cos x} (x) = 0$$

$$0 - \frac{x \cos x}{x \sin x + \cos x} + \frac{x \cos x}{x \sin x + \cos x} = 0 \rightarrow \mathbf{0 = 0}$$

QED

Assuming that there a function $u(x)$ such that $y_2(x) = y_1(x)u(x)$ is a second solution to the ODE:

$$y_2(x) = xu(x)$$

Substituting into ODE:

$$y_2 = xu$$

$$y_2' = u + xu'$$

$$y_2'' = u' + u' + xu'' = 2u' + xu''$$

$$(2u' + xu'') - \frac{x \cos x}{x \sin x + \cos x} (u + xu') + \frac{\cos x}{x \sin x + \cos x} (xu) = 0$$

$$xu'' - xu' \left(\frac{x \cos x}{x \sin x + \cos x} - \frac{2}{x} \right) + u \left(\frac{x \cos x}{x \sin x + \cos x} - \frac{x \cos x}{x \sin x + \cos x} \right) = 0$$

$$xu'' - xu' \left(\frac{x \cos x}{x \sin x + \cos x} - \frac{2}{x} \right) = 0$$

$$v = u'$$

$$xv' - xv \left(\frac{x \cos x}{x \sin x + \cos x} - \frac{2}{x} \right) = 0$$

$$v' - v \left(\frac{x \cos x}{x \sin x + \cos x} - \frac{2}{x} \right) = 0$$

$$\int \frac{v'}{v} dx = \int \frac{x \cos x}{x \sin x + \cos x} - \frac{2}{x} dx$$

$$\ln v = \ln(x \sin x + \cos x) - 2 \ln x$$

$$\ln v = \ln \frac{x \sin x + \cos x}{x^2}$$

$$v = \frac{x \sin x + \cos x}{x^2}$$

$$u = \int \frac{x \sin x + \cos x}{x^2} dx$$

$$u = \int \frac{\sin x}{x} + \frac{\cos x}{x^2} dx$$

$$u = \int \frac{\sin x}{x} dx + \int \frac{1}{x^2} \cdot \cos x dx$$

$$f = \cos x$$

$$f' = -\sin x$$

$$g' = \frac{1}{x^2}$$

$$g = -\frac{1}{x}$$

$$u = \int \frac{\sin x}{x} dx + fg - \int f' g dx$$

$$u = \int \frac{\sin x}{x} dx + \left(-\frac{1}{x} \cdot \cos x \right) - \int (-\sin x) \cdot \left(-\frac{1}{x} \right) dx$$

$$u = \int \frac{\sin x}{x} dx - \int \frac{\sin x}{x} dx - \frac{\cos x}{x}$$

$$u = -\frac{\cos x}{x}$$

$$y_2 = xu$$

$$y_2 = x\left(-\frac{\cos x}{x}\right)$$

$$\mathbf{y_2 = -\cos x}$$

(3)

The general solution is:

$$y(x) = c_1x + c_2 \cos x$$

Using the conditions $y(0) = 1$ and $y'(0) = 1$:

$$y(0) = c_1(0) + c_2 \cos 0 = 1$$

$$c_2 = 1$$

$$y'(x) = c_1 - c_2 \sin x$$

$$y'(0) = c_1 - c_2 \sin 0 = 1$$

$$c_1 = 1$$

$$\mathbf{y(x) = x + \cos x}$$

3.

(1)

$$y'' + y = \frac{1}{\cos^3 t}$$

The complementary solution $y_c(t)$:

Solving the homogenous ODE:

$$y'' + y = 0$$

Writing the characteristic equation:

$$r^2 + 1 = 0$$

$$r = 0 \pm i$$

Then the complementary solution is:

$$y_c(t) = c_1 e^{-0t} \cos t + c_2 e^{-0t} \sin t$$

$$y_c(t) = c_1 \cos t + c_2 \sin t$$

The particular solution $y_p(t)$:

Using the method of variation of parameters:

Let's assume that the particular solution is of the form:

$$y_p(t) = u_1(t) \cos t + u_2(t) \sin t$$

where $u_1(t)$ and $u_2(t)$ are to be found.

$$f(t) = \sec^3 t$$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t - (-\sin^2 t) = 1$$

$$u_1(t) = - \int \frac{f(t)y_2(t)}{W(t)} dt$$

$$u_1(t) = - \int \frac{\sec^3 t \cdot \sin t}{1} dt$$

$$u_1(t) = - \int \frac{\sec^3 t \cdot (\tan t \cdot \cos t)}{1} dt$$

$$u_1(t) = - \int \sec^2 t \cdot \tan t dt$$

$$v = \tan t$$

$$\frac{dv}{dt} = \sec^2 t$$

$$dt = \frac{dv}{\sec^2 t}$$

$$u_1(t) = - \int v dv$$

$$u_1(t) = -\frac{1}{2}v^2 = -\frac{1}{2}\tan^2 t$$

$$u_1(t) = \tan t$$

$$u_2(t) = \int \frac{f(t)y_1(t)}{W(t)} dt$$

$$u_2(t) = \int \frac{\sec^3 t \cdot \cos t}{1} dt$$

$$u_2(t) = \int \sec^2 t dt$$

$$u_2(t) = \tan t$$

$$y_p(t) = \tan t \sin t - \frac{1}{2} \tan^2 t \cos t$$

$$y_p(t) = \frac{1}{2} \tan t (2 \sin t - \tan t \cos t)$$

$$y_p(t) = \frac{1}{2} \tan t (2 \sin t - \sin t)$$

$$y_p(t) = \frac{1}{2} \tan t \sin t$$

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{2} \tan t \sin t$$

$$\mathbf{y(t) = c_1 \cos t + (c_2 + \frac{1}{2} \tan t) \sin t}$$

(2)

Using the conditions $y(0) = 0$ and $y'(0) = 0$:

$$y(0) = c_1 \cos 0 + \left(c_2 + \frac{1}{2} \tan 0\right) \sin 0 = 0$$

$$c_1 = 0$$

$$y'(t) = -c_1 \sin t + \left(\frac{1}{2} \sec^2 t\right) \sin t + \left(c_2 + \frac{1}{2} \tan t\right) \cos t$$

$$y'(0) = -c_1 \sin 0 + \left(\frac{1}{2} \sec^2 0\right) \sin 0 + \left(c_2 + \frac{1}{2} \tan 0\right) \cos 0 = 0$$

$$c_2 = 0$$

$$y(t) = 0 \cos t + \left(0 + \frac{1}{2} \tan t\right) \sin t$$

$$\mathbf{y(t) = \frac{1}{2} \tan t \sin t}$$